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Optimal Control via Mathematical Programming

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Introduction

OPTIMAL control problems with bounded controls reduce to two-point boundary-value problems which are difficult to handle by conventional methods of calculus of variations. Pontryagin's maximum principle and Bellman's dynamic programming are the other methods for solving such problems. In this paper the optimal control problem is transformed into one of mathematical programming by a Raleigh-Ritz-type procedure. This is essentially a penalty function approach which penalizes the given performance index for not satisfying the differential constraints. The resulting mathematical programming problem is then solved by SWIFT—Sequential Weightage Increasing Factor Technique developed by the authors.[1] The method is illustrated by obtaining the thrust angle history of a probe for an Earth-to-Mars orbit transfer in minimum time. Good comparison of the time obtained by this method with the actual time taken by the Viking-2 spacecraft is extremely interesting. Other methods[2,3] do not show such a good comparison.

Problem Formulation

Let the dynamics of the system be described by a set of ordinary differential equations given by

$$\dot{X}=f(X,V,t) \qquad X_i(t_0)=X_{i0} \qquad (i=1,2,\ldots,K) \tag{1}$$

where $X(t)$ is a state vector and $V(t)$ is a control vector. Given the initial conditions for the states $X_i(t_0)$, the problem is to find the control vector $V(t_0)$, $t_0 \leq t \leq t_f$ which transfers the system from $X(t_0)$ to $X(t_f)$ such that

$$J=\int_{t_0}^{t_f} g(X,V,t)\,\mathrm{d}t \tag{2}$$

is minimum. In the following, t_0 is assumed equal to zero.

The states and the controls are written in terms of a Raleigh-Ritz type of expansion such that

$$X_i(t)=X_i(0)+\left\{\frac{X_i(t_f)-X_i(0)}{t_f}\right\}t+\sum_{m=1}^{M} a_{im}\sin\frac{m\pi t}{t_f} \qquad (i=1,2,3,\ldots,K) \tag{3}$$

$$V_j(t)=V_j(0)+\left\{\frac{V_j(t_f)-V_j(0)}{t_f}\right\}t+\sum_{m=1}^{M} b_{jm}\sin\frac{m\pi t}{t_f} \qquad (j=1,2,3,\ldots,L) \tag{4}$$

One notes that Eqs. (3) and (4) satisfy the boundary conditions exactly. Substituting Eqs. (3) and (4) in Eq. (1) we

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Table 1 Minimum time and dynamic error for various values of N and M

	$M=3$		$M=4$	
N	t_f	E	t_f	E
4	5.83	8.1×10^{-8}	2.53	8.33×10^{-3}
6	3.23	2.8×10^{-3}	4.06	3.47×10^{-5}
8	2.97	9.3×10^{-3}	3.60	1.28×10^{-3}
12	1.33	2.69×10^{-1}	2.95	9.14×10^{-3}

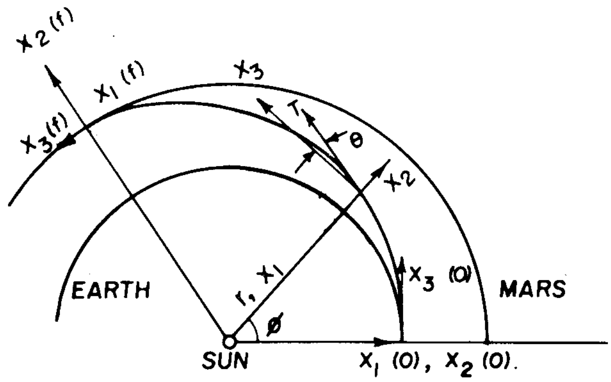

Fig. 1 Kinematics for Earth-to-Mars transfer orbit.

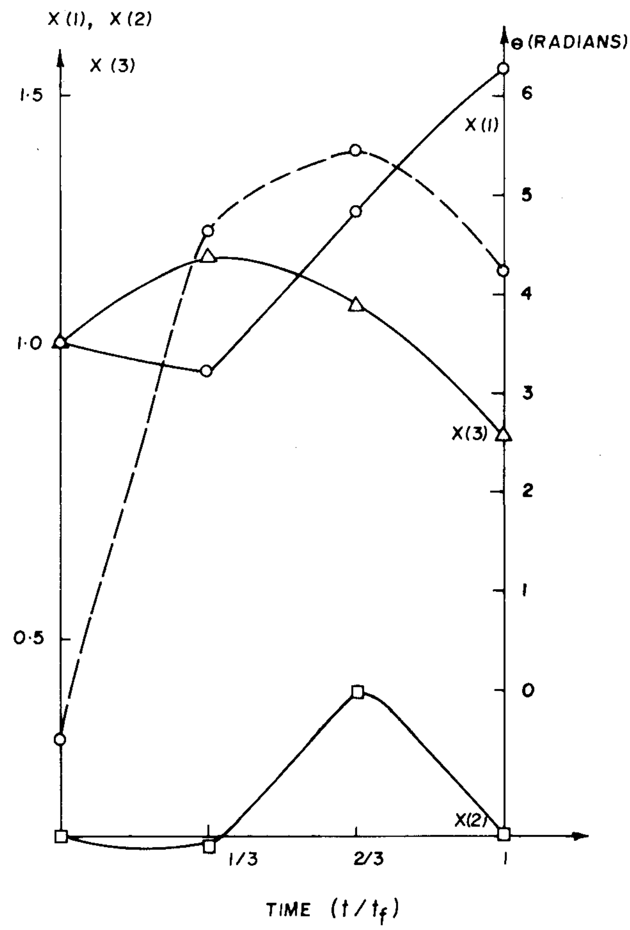

Fig. 2 Optimal state and control histories ($M=3$, $N=4$, $(N-1)\Delta t=t_f$).

define the dynamic error E as

$$E=\int_0^{t_f}\left\{\sum_{i=1}^{K}[f_i(X,V,t)-\dot{X}_i(t)]^2\right\}dt \qquad (5)$$

The optimal control problem now reduces to minimizing J given by Eq. (2) under the constraint

$$E=0 \qquad (6)$$

in order to obtain a_{im} and b_{jm} ($i=1,2,...,K$; $j=1,2,...,L$; $m=1,2,...,M$). This can be achieved by a multivariable search method such as SWIFT.[1] In the problem of minimum time $g=1$ and $J=t_f$.

The first step in the iterative procedure is to choose M, the number of functions to be assumed for an adequate expansion in Eqs. (3) and (4), and N, the number of discrete points $t_0,...,t_f=(t_{N-1})$ in the interval $(0,t_f)$. Each time interval is given by $\Delta t=t_{i+1}-t_i$ so that the total time $t_f=N\Delta t$. Choosing Δt, a_{im}, b_{jm} as the parameters for optimization, one can use Eqs. (3-6) repeatedly before a preset convergence criterion is satisfied.

Example

The above procedure is applied to obtain the thrust angle history for transferring a space probe from Earth's orbit to Mars' orbit. The dynamic equations of a space vehicle are given in Ref. 2 and illustrated in Fig. 1. The nondimensional equations can be shown to be

$$\dot{X}_1=X_2 \qquad \dot{X}_2=\frac{X_3^2}{X_1}-\frac{1}{X_1^2}+\frac{T\sin\theta}{m}$$

$$\dot{X}_3=-\frac{X_2X_3}{X_1}+\frac{T\cos\theta}{m} \qquad (7)$$

where $m=1-\dot{m}t$ and where mass and length are measured by the initial values m_0, r_0 and time is measured by $t_0=r_0/V_0$, where V_0 is the initial velocity. θ is measured in radians; thrust is measured by $T_0=m_0r_0/t_0^2$.

The nondimensional burning rate and thrust are assumed constant such that $\dot{m}=0.0749$ and $T=0.1405$. These values are appropriate for a 35 kW propulsion system. The weight at escape is 1500 lb, specific impulse is 5700 s, and rate of burning is 1.95 lb/day. These are very close to the parametric values of the vehicle to be used with the Saturn rocket and SNAP-8 reactor.[4] Boundary conditions turn out to be: $X_1(0)=1.0$, $X_2(0)=0.0$, $X_3(0)=1.0$; $X_1(t_f)=1.525$, $X_2(t_f)=0$ and $X_3(t_f)=0.8098$. In this problem θ is the control variable.

Table 1 gives the minimum time t_f, the dynamic error E for various values of N and M.

We note that the first solution which gives $t_f=5.83$ is the best solution as the dynamic error constraint ($E=8.1\times10^{-8}$) is adequately satisfied. The time taken for the space probe is 338 days ($=5.83\times365/2\pi$). This value compares extremely well with that of Viking-2 which took off on Sept. 9, 1975 and swung into Mars' orbit on Aug. 7, 1976 (about 333 days). Figure 2 gives the state and control time histories. To the authors' knowledge a solution for this particular problem with conventional methods of solving the TPBVP is not available for comparison. Taylor et al.[2] give the minimum time as 3.39 with $E=10^{-4}$ while Williamson[3] gives $t_f=3.4199$. These values and also the other cases of Table 1 do not give realistic time.

References

[1] Sheela, B.V. and Ramamoorthy, P., "SWIFT—A New Constrained Optimization Technique," *Computer Methods in Applied Mechanics and Engineering,* Vol. 6, Dec. 1975, p. 309.

[2] Taylor, J.M. and Constantinides, C.T., "Optimization: Application of the Epsilon Method," *IEEE Transactions on Automatic Control,* Vol. AC 17, Feb. 1972, p. 128.

[3] Williamson, W.E., "Use of Polynomial Approximations to Calculate Suboptimal Controls," *AIAA Journal,* Vol. 9, Nov. 1971, p. 2271.

Technical Comments

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Comment on "Sequential Estimation Algorithm Using a Continuous UDU^T Covariance Factorization"

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Tapley and Peters[1] propose replacing the discrete-time U-D covariance factor Gram-Schmidt time update described in Refs. 2 and 3. Their approach is to modify an idea due to Andrews,[4] and they present a set of U-D element differential equations. To justify their results, they consider a LANDSAT-D satellite navigation application. Their comparison of results and the conclusions they draw do not, however, adequately describe the relative merits of differential equation U-D element propagation and discrete time propagation. Our purpose is to briefly highlight key comparison items that were not discussed in Ref. 1, and to clarify some misleading statements made therein.

Problem Introduction

To best understand the distinction between the continuous and discrete-time algorithm we consider the linear continuous time model

$$\dot{x}=Ax+B_c\xi \qquad x(t_0)\in N(\bar{x}_0,P_0) \qquad (1)$$

where A and B_c may be time dependent, $\xi(t)$ is white process noise with diagonal intensity $Q_c=Q_c(t)$, and the symbol $N(\bar{x}_0,P_0)$ indicates that $x(t_0)$ is normal, with mean $\bar{x}_0$ and covariance P_0. For simplicity of exposition the usual assumption made is that $\xi(t)$ is independent of $x(t_0)$. Because Eq. (1) is linear, one can express the solution to the differential equation as

$$x(t_{j+1})=\Phi(t_{j+1},t_j)x(t_j)+B_Dw_j \qquad (2)$$

where $\{B_Dw_j\}$ is a white noise sequence and $\Phi(t_{j+1},t_j)$ is the solution to the linear differential equation

$$\frac{d}{dt}\Phi(t,t_j)=A\Phi(t,t_j) \qquad (\Phi(t,t_j)=I \text{ for } t=t_j) \qquad (3)$$

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